REAL NUMBERS

1
CHAPTER

CONTENTS

➤ NUMBER SYSTEM

❖ Natural Numbers :

The simplest numbers are 1, 2, 3, 4....... the numbers being used in counting. These are called natural numbers.

❖ Whole numbers :

The natural numbers along with the zero form the set of whole numbers i.e. numbers 0, 1, 2, 3, 4 are whole numbers. W = {0, 1, 2, 3, 4....}

❖ Integers :

The natural numbers, their negatives and zero make up the integers.

$Z = \{....-4, -3, -2, -1, 0, 1, 2, 3, 4,....\}$

The set of integers contains positive numbers, negative numbers and zero.

❖ Rational Number :

(i) A rational number is a number which can be put in the form $\frac{p}{q}$, where p and q are both integers and $q \neq 0$.

(ii) A rational number is either a terminating or non-terminating and recurring (repeating) decimal.

(iii) A rational number may be positive, negative or zero.

❖ Complex numbers :

Complex numbers are imaginary numbers of the form a + ib, where a and b are real numbers and $i = \sqrt{-1}$, which is an imaginary number.

❖ Factors :

A number is a factor of another, if the former exactly divides the latter without leaving a remainder (remainder is zero) 3 and 5 are factors of 12 and 25 respectively.

- **Multiples :**

  A multiple is a number which is exactly divisible by another, 36 is a multiple of 2, 3, 4, 9 and 12.

- **Even Numbers :**

  Integers which are multiples of 2 are even number (i.e.) 2,4, 6, 8............... are even numbers.

- **Odd numbers :**

  Integers which are not multiples of 2 are odd numbers.

- **Prime and composite Numbers :**

  All natural number which cannot be divided by any number other than 1 and itself is called a prime number. By convention, 1 is not a prime number. 2, 3, 5, 7, 11, 13, 17 .... are prime numbers. Numbers which are not prime are called composite numbers.

- **The Absolute Value (or modulus) of a real Number :**

  If a is a real number, modulus a is written as $|a|$; $|a|$ is always positive or zero.It means positive value of 'a' whether a is positive or negative

  $|3| = 3$ and $|0| = 0$, Hence $|a| = a$; if $a = 0$ or $a > 0$ (i.e.) $a \geq 0$

  $|-3| = 3 = -(-3)$. Hence $|a| = -a$ when $a < 0$

  Hence, $|a| = a$, if $a > 0$; $|a| = -a$, if $a < 0$

- **Irrational number :**

  (i) A number is irrational if and only if its decimal representation is non-terminating and non-repeating. e.g. $\sqrt{2}$, $\sqrt{3}$, π................ etc.

  (ii) Rational number and irrational number taken together form the set of real numbers.

  (iii) If a and b are two real numbers, then either (i) $a > b$ or (ii) $a = b$ or (iii) $a < b$

  (iv) Negative of an irrational number is an irrational number.

  (v) The sum of a rational number with an irrational number is always irrational.

  (vi) The product of a non-zero rational number with an irrational number is always an irrational number.

  (vii) The sum of two irrational numbers is not always an irrational number.

  (viii) The product of two irrational numbers is not always an irrational number.

  In division for all rationals of the form $\frac{p}{q}$ $(q \neq 0)$, p & q are integers, two things can happen either the remainder becomes zero or never becomes zero.

  Type (1) : Eg : $\frac{7}{8} = 0.875$

$$
\begin{array}{r|l}
8\,)\,70 & (0.875 \\
64 & \\
\hline
60 & \\
56 & \\
\hline
40 & \\
40 & \\
\hline
\times &
\end{array}
$$

  This decimal expansion 0.875 is called **terminating**.

$\therefore$ If remainder is zero then decimal expansion ends (terminates) after finite number of steps. These decimal expansion of such numbers terminating.

Type (2) :

Eg : $\frac{1}{3} = 0.333\ldots\ldots\ldots$

$= 0.\overline{3}$

$$\begin{array}{r|l} 3 & 10 \;(0.33\ldots\ldots \\ & \underline{9} \\ & 10 \\ & \underline{9} \\ & 1\ldots\ldots \end{array}$$

or $\frac{1}{7} = 0.142857142857\ldots..$

$= 0.\overline{142857}$

$$\begin{array}{r|l} 7 & 10 \;(0.14285\ldots. \\ & \underline{7} \\ & 30 \\ & \underline{28} \\ & 20 \\ & \underline{14} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{35} \\ & 50 \\ & \underline{49} \\ & 1\ldots. \end{array}$$

In both examples remainder is never becomes zero so the decimal expansion is never ends after some or infinite steps of division. These type of decimal expansions are called **non terminating.**

In above examples, after I^{st} step & 6 steps of division (respectively) we get remainder equal to dividend so decimal expansion is repeating (recurring).

So these are called **non terminating recurring decimal expansions**.

Both the above types (1 & 2) are rational numbers.

Types (3) :

Eg : The decimal expansion 0.327172398……is not ends any where, also there is no arrangement of digits (not repeating) so these are called **non terminating not recurring**.

These numbers are called **irrational numbers**.

Eg. :

0.1279312793	rational	terminating
0.1279312793….	rational	non terminating
or $0.\overline{12793}$		& recurring
0.32777	rational	terminating
$0.32\overline{7}$ or	rational	non terminating

0.32777…….		& recurring
0.5361279	rational	terminating
0.3712854043….	irrational	non terminating
		non recurring
0.10100100010000	rational	terminating
0.10100100010000….	irrational	non terminating
		non recurring.

Real Numbers

- Rational no.
  - If remainder = 0 → Terminating & Non repeating
    Eg : $\frac{18}{5} = 3.6$
  - If remainder ≠ 0 & rem. = devidend → Non terminating & repeating (recurring)
    Eg : $\frac{1}{3} = 0.33\ldots\ldots$ $= 0.\overline{3}$
- Irrational no.
  - If remainder ≠ 0 & rem. ≠ any devidend → Non terminating non repeating
    Eg : 0.671234…..
    Eg : 1.343634003908…...

❖ EXAMPLES ❖

Ex.1 Insert a rational and an irrational number between 2 and 3.

Sol. If a and b are two positive rational numbers such that ab is not a perfect square of a rational number, then $\sqrt{ab}$ is an irrational number lying between a and b. Also, if a,b are rational numbers, then $\frac{a+b}{2}$ is a rational number between them.

$\therefore$ A rational number between 2 and 3 is

$$\frac{2+3}{2} = 2.5$$

An irrational number between 2 and 3 is

$$\sqrt{2\times 3} = \sqrt{6}$$

Ex.2 Find two irrational numbers between 2 and 2.5.

Sol. If a and b are two distinct positive rational numbers such that ab is not a perfect square of a rational number, then $\sqrt{ab}$ is an irrational number lying between a and b.

$\therefore$ Irrational number between 2 and 2.5 is

$$\sqrt{2\times 2.5} = \sqrt{5}$$

Similarly, irrational number between 2 and $\sqrt{5}$ is $\sqrt{2\times\sqrt{5}}$

So, required numbers are $\sqrt{5}$ and $\sqrt{2\times\sqrt{5}}$.

Ex.3 Find two irrational numbers lying between $\sqrt{2}$ and $\sqrt{3}$.

Sol. We know that, if a and b are two distinct positive irrational numbers, then $\sqrt{ab}$ is an irrational number lying between a and b.

$\therefore$ Irrational number between $\sqrt{2}$ and $\sqrt{3}$ is $\sqrt{\sqrt{2}\times\sqrt{3}} = \sqrt{\sqrt{6}} = 6^{1/4}$

Irrational number between $\sqrt{2}$ and $6^{1/4}$ is $\sqrt{\sqrt{2}\times 6^{1/4}} = 2^{1/4}\times 6^{1/8}$.

Hence required irrational number are $6^{1/4}$ and $2^{1/4}\times 6^{1/8}$.

Ex.4 Find two irrational numbers between 0.12 and 0.13.

Sol. Let a = 0.12 and b = 0.13. Clearly, a and b are rational numbers such that a < b.

We observe that the number a and b have a 1 in the first place of decimal. But in the second place of decimal a has a 2 and b has 3. So, we consider the numbers

c = 0.1201001000100001 ......

and, d = 0.12101001000100001.......

Clearly, c and d are irrational numbers such that a < c < d < b.

Theorem : Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.

Proof : Let the prime factorisation of a be as follows :

$a = p_1p_2.....p_n$, where $p_1, p_2, ...p_n$ are primes, not necessarily distinct.

Therefore,

$a^2 = (p_1p_2.....p_n)(p_1p_2 ..... p_n) = p_1^2p_2^2.....p_n^2$.

Now, we are given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1, p_2, \ldots, p_n$. So p is one of $p_1, p_2, \ldots\ldots, p_n$.

Now, since $a = p_1\,p_2 \ldots\ldots p_n$, p divides a.

We are now ready to give a proof that $\sqrt{2}$ is irrational.

The proof is based on a technique called 'proof by contradiction'.

Ex.5 Prove that

(i) $\sqrt{2}$ is irrational number

(ii) $\sqrt{3}$ is irrational number

Similarly $\sqrt{5}, \sqrt{7}, \sqrt{11}$ …... are irrational numbers.

Sol. (i) Let us assume, to the contrary, that $\sqrt{2}$ is rational.

So, we can find integers r and s ($\neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s not having a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are coprime.

So, $b\sqrt{2} = a$.

Squaring on both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2. Now, by Theorem it following that 2 divides a.

So, we can write $a = 2c$ for some integer c.

Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using Theorem with $p = 2$).

Therefore, a and b have at least 2 as a common factor.

But this contradicts the fact that a and b have no common factors other than 1.

This contradiction has arisen because of our incorrect assumption that $\sqrt{2}$ is rational.

So, we conclude that $\sqrt{2}$ is irrational.

(ii) Let us assume, to contrary, that $\sqrt{3}$ is rational. That is, we can find integers a and b ($\neq 0$) such that $\sqrt{3} = \frac{a}{b}$.

Suppose a and b not having a common factor other than 1, then we can divide by the common factor, and assume that a and b are coprime.

So, $b\sqrt{3} = a$.

Squaring on both sides, and rearranging, we get $3b^2 = a^2$.

Therefore, a^2 is divisible by 3, and by Theorem, it follows that a is also divisible by 3.

So, we can write $a = 3c$ for some integer c.

Substituting for a, we get $3b^2 = 9c^2$, that is, $b^2 = 3c^2$.

This means that b^2 is divisible by 3, and so b is also divisible by 3 (using Theorem with $p = 3$).

Therefore, a and b have at least 3 as a common factor.

But this contradicts the fact that a and b are coprime.

This contradicts the fact that a and b are coprime.

This contradiction has arisen because of our incorrect assumption that $\sqrt{3}$ is rational.

So, we conclude that $\sqrt{3}$ is irrational.

Ex.6 Prove that $7 - \sqrt{3}$ is irrational

Sol. **Method I :**

Let $7 - \sqrt{3}$ is rational number

$\therefore\ 7 - \sqrt{3} = \frac{p}{q}$ (p, q are integers, $q \neq 0$)

$\therefore\ 7 - \frac{p}{q} = \sqrt{3}$

$\Rightarrow\ \sqrt{3} = \frac{7q - p}{q}$

Here p, q are integers

$\therefore \ \dfrac{7q-p}{q}$ is also integer

$\therefore$ LHS = $\sqrt{3}$ is also integer but this is contradiction that $\sqrt{3}$ is irrational so our assumption is wrong that $(7-\sqrt{3})$ is rational

$\therefore \ 7-\sqrt{3}$ is irrational proved.

Method II :

Let $7-\sqrt{3}$ is rational

we know sum or difference of two rationals is also rational

$\therefore \ 7-(7-\sqrt{3})$

= $\sqrt{3}$ = rational

but this is contradiction that $\sqrt{3}$ is irrational

$\therefore \ (7-\sqrt{3})$ is irrational proved.

Ex.7 Prove that :

(i) $\dfrac{\sqrt{5}}{3}$ (ii) $2\sqrt{7}$ are irrationals

Sol. (i) Let $\dfrac{\sqrt{5}}{3}$ is rational

$\therefore \ 3\left(\dfrac{\sqrt{5}}{3}\right) = \sqrt{5}$ is rational

($\because$ product of two rationals is also rational)

but this is contradiction that $\sqrt{5}$ is irrational

$\therefore \ \dfrac{\sqrt{5}}{3}$ is irrational proved.

(ii) Let $2\sqrt{7}$ is rational

$\therefore \ (2\sqrt{7})\times\dfrac{1}{2} = \sqrt{7}$

($\because$ division of two rational no. is also rational)

$\therefore \ \sqrt{7}$ is rational

but this is contradiction that $\sqrt{7}$ is irrational

$\therefore \ 2\sqrt{7}$ is irrational

proved

Theorem 1 :

Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$, where p and q are coprime and the prime factorization of q is of the form $2^n 5^m$, where n, m are non-negative integers.

(A) Numbers are terminating (remainder = zero)

Eg : $\frac{32}{125} = \frac{2^5}{5^3} = \frac{2^8}{(2\times5)^3} = \frac{256}{10^3} = 0.256$

Eg : $\frac{9}{25} = \frac{9\times2^2}{5^2\times2^2} = \frac{36}{(2\times5)^2} = \frac{36}{(10)^2} = 0.36$

So we can convert a rational number of the form $\frac{p}{q}$, where q is of the form $2^n 5^m$ to an equivalent rational number of the form $\frac{a}{b}$ where b is a power of 10. These are terminates.

OR

Theorem 2 :

Let x = $\frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form $2^n\, 5^m$, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

(B) Non terminating & recurring

Eg : $\frac{1}{7} = 0.\overline{142857} = 0.142857142857.....$

Since denominator 7 is not of the form $2^n\, 5^m$ so we zero (0) will not show up as a remainder.

Theorem 3 :

Let x = $\frac{p}{q}$ be a rational number, such that the prime factorization of q is not of the form $2^n 5^m$, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).

From the discussion above, we can conclude that the decimal expansion of every rational number is either terminating or non-terminating repeating.

Eg : From given rational numbers check terminating or non terminating

(1) $\frac{13}{3125} = \frac{13}{(5)^5} = \frac{13\times2^5}{2^5\times5^5} = \frac{(13\times32)}{(10)^5}$

= terminating

(2) $\frac{17}{8} = \frac{17}{2^3} = \frac{17\times5^3}{(2\times5)^3} = \frac{17\times125}{(10)^3}$

= terminating

(3) $\frac{64}{455} = \frac{2^6}{5 \times 7 \times 13}$ ($\because$ we can not remove 7 & 13 from dinominator) non-terminating repeating ($\because$ no. is rational $\therefore$ it is always repeating or recurring)

(4) $\frac{15}{1600} = \frac{3 \times 5}{2^4 \times 10^2} = \frac{3 \times 5^5}{(2 \times 5)^4 \times 10^2} = \frac{3 \times 5^5}{10^6}$

= terminating

(5) $\frac{29}{343} = \frac{29}{(7)^3}$ = non terminating

(6) $\frac{23}{2^3 5^2} = \frac{23 \times 5}{(2 \times 5)^3} = \frac{23 \times 5}{(10)^3}$

= terminating

(7) $\frac{129}{2^5 \times 5^7 \times 7^5} = \frac{3 \times 43 \times 2^2}{(2 \times 5)^7 \times 7^5}$

= non terminating ($\because$ 7 cannot remove from denominator)

(8) $\frac{6}{15} = \frac{2 \times 3}{5 \times 3} = \frac{2}{5} = \frac{2 \times 2}{10}$

= terminating

(9) $\frac{35}{50} = \frac{35 \times 2}{100}$ = terminating

(10) $\frac{77}{210} = \frac{7 \times 11}{7 \times 30} = \frac{7 \times 11}{7 \times 2 \times 5 \times 3}$

= non terminating

➢ EUCLID'S DIVISION LEMMA OR EUCLID'S DIVISION ALGORITHM

For any two positive integers **a** and **b**, there exist unique integers q and r satisfying a = bq + r, where $0 \le r < b$.

For Example

(i) Consider number 23 and 5, then:

$23 = 5 \times 4 + 3$

Comparing with a = bq + r; we get:

a = 23, b = 5, q = 4, r = 3

and $0 \le r < b$ (as $0 \le 3 < 5$).

(ii) Consider positive integers 18 and 4.

$18 = 4 \times 4 + 2$

$\Rightarrow$ For 18 (= a) and 4(= b) we have q = 4,

r = 2 and $0 \le r < b$.

In the relation a = bq + r, where $0 \le r < b$ is nothing but a statement of the long division of number a by number b in which q is the quotient obtained and r is the remainder.

Thus, dividend = divisor × quotient + remainder ⇒ $a = bq + r$

◈ H.C.F. (Highest Common Factor)

The H.C.F. of two or more positive integers is the largest positive integer that divides each given positive number completely.

i.e., if positive integer **d** divides two positive integers **a** and **b** then the H.C.F. of **a** and **b** is **d**.

For Example

(i) 14 is the largest positive integer that divides 28 and 70 completely; therefore H.C.F. of 28 and 70 is 14.

(ii) H.C.F. of 75, 125 and 200 is 25 as 25 divides each of 75, 125 and 200 completely and so on.

◈ Using Euclid's Division Lemma For Finding H.C.F.

Consider positive integers 418 and 33.

Step-1

Taking bigger number (418) as **a** and smaller number (33) as **b**

express the numbers as $a = bq + r$

⇒ $418 = 33 \times 12 + 22$

Step-2

Now taking the divisor 33 and remainder 22; apply the Euclid's division algorithm to get:

$33 = 22 \times 1 + 11$ [Expressing as $a = bq + r$]

Step-3

Again with new divisor 22 and new remainder 11; apply the Euclid's division algorithm to get:

$22 = 11 \times 2 + 0$

Step-4

Since, the remainder = 0 so we cannot proceed further.

Step-5

The last divisor is 11 and we say H.C.F. of 418 and 33 = 11

Verification :

(i) Using factor method:

∴ Factors of 418 = 1, 2, 11, 19, 22, 38, 209 and 418 and,

Factor of 33 = 1, 3, 11 and 33.

Common factors = 1 and 11

⇒ Highest common factor = 11 i.e., H.C.F. = 11

(ii) Using prime factor method:

Prime factors of 418 = 2, 11 and 19.

Prime factors of 33 = 3 and 11.

∴ **H.C.F.** = Product of all common prime factors = 11. For any two positive integers a and b which can be expressed as $a = bq + r$, where $0 \le r < b$, the, H.C.F. of (a, b) = H.C.F. of (q, r) and so on. For number 418 and 33

$418 = 33 \times 12 + 22$

$33 = 22 \times 1 + 11$

and $22 = 11 \times 2 + 0$

$\Rightarrow$ H.C.F. of (418, 33) = H.C.F. of (33, 22)

= H.C.F. of (22, 11) = 11.

❖EXAMPLES ❖

Ex.8 Using Euclid's division algorithm, find the H.C.F. of **[NCERT]**

(i) 135 and 225 (ii) 196 and 38220

(iii) 867 and 255

Sol.(i) Starting with the larger number i.e., 225, we get:

$225 = 135 \times 1 + 90$

Now taking divisor 135 and remainder 90, we get $135 = 90 \times 1 + 45$

Further taking divisor 90 and remainder 45, we get $90 = 45 \times 2 + 0$

$\therefore$ **Required H.C.F. = 45** **(Ans.)**

(ii) Starting with larger number 38220, we get:

$38220 = 196 \times 195 + 0$

Since, the remainder is 0

$\Rightarrow$ **H.C.F. = 196** **(Ans.)**

(iii) Given number are 867 and 255

$\Rightarrow$ $867 = 255 \times 3 + 102$ **(Step-1)**

$255 = 102 \times 2 + 51$ **(Step-2)**

$102 = 51 \times 2 + 0$ **(Step-3)**

$\Rightarrow$ **H.C.F. = 51** **(Ans.)**

Ex.9 Show that every positive integer is of the form 2q and that every positive odd integer is of the from 2q + 1, where q is some integer.

Sol. According to Euclid's division lemma, if a and b are two positive integers such that a is greater than b; then these two integers can be expressed as

$a = bq + r$; where $0 \le r < b$

Now consider

$b = 2$; then $a = bq + r$ will reduce to

$a = 2q + r$; where $0 \le r < 2$,

i.e., $r = 0$ or $r = 1$

If $r = 0$, $a = 2q + r \Rightarrow a = 2q$

i.e., a is even

and, if $r = 1$, $a = 2q + r \Rightarrow a = 2q + 1$

i.e., a is add;

as if the integer is not even; it will be odd.

Since, a is taken to be any positive integer so it is applicable to the every positive integer that when it can be expressed as

$a = 2q$

$\therefore$ **a** is even and when it can expressed as

$a = 2q + 1$; a is odd.

Hence the required result.

Ex.10 Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where q is some integer.

Sol. Let **a** is **b** be two positive integers in which **a** is greater than **b**. According to Euclid's division algorithm; **a** and **b** can be expressed as

$a = bq + r$, where q is quotient and r is remainder and $0 \le r < b$.

Taking $b = 4$, we get: $a = 4q + r$,

where $0 \le r < 4$ i.e., $r = 0, 1, 2$ or 3

$\mathbf{r = 0} \Rightarrow a = 4q$, which is divisible by 2 and so is **even**.

$\mathbf{r = 1} \Rightarrow a = 4q + 1$, which is not divisible by 2 and so is **odd**.

$\mathbf{r = 2} \Rightarrow q = 4q + 2$, which is divisible by 2 and so is **even**.

and $\mathbf{r = 3} \Rightarrow q = 4q + 3$, which is not divisible by 2 and so is **odd**.

$\therefore$ **Any positive odd integer is of the form**

$\mathbf{4q + 1}$ **or** $\mathbf{4q + 3}$**;** where q is an integer.

Hence the required result.

Ex.11 Show that one and only one out of n; $n + 2$ or $n + 4$ is divisible by 3, where **n** is any positive integer.

Sol. Consider any two positive integers **a** and **b** such that **a** is greater than **b**, then according to Euclid's division algorithm:

$a = bq + r$; where q and r are positive integers and $0 \le r < b$

Let $a = n$ and $b = 3$, then

$a = bq + r \Rightarrow n = 3q + r$; where $0 \le r < 3$.

$\mathbf{r = 0} \Rightarrow n = 3q + 0 = 3q$

$\mathbf{r = 1} \Rightarrow n = 3q + 1$ and $\mathbf{r = 2} \Rightarrow n = 3q + 2$

If $n = 3q$; **n is divisible by 3**

If $n = 3q + 1$; then $n + 2 = 3q + 1 + 2$

$= 3q + 3$; which is divisible by 3

$\Rightarrow$ **n + 2 is divisible by 3**

If $n = 3q + 2$; then $n + 4 = 3q + 2 + 4$

$= 3q + 6$; which is divisible by 3

$\Rightarrow$ **n + 4 is divisible by 3**

Hence, if **n** is any positive integer, then one and only one out of n, $n + 2$ or $n + 4$ is divisible by 3.

Hence the required result.

Ex.12 Show that any positive integer which is of the form $6q + 1$ or $6q + 3$ or $6q + 5$ is odd, where q is some integer.

Sol. If **a** and **b** are two positive integers such that **a** is greater than **b**; then according to Euclid's division algorithm; we have

$a = bq + r$; where q and r are positive integers and $0 \le r < b$.

Let $b = 6$, then

$a = bq + r \Rightarrow$ **$a = 6q + r$**; where $0 \le r < 6$.

When $r = 0 \Rightarrow a = 6q + 0 = 6q$;

which is even integer

When $r = 1 \Rightarrow a = 6q + 1$

which is odd integer

When $r = 2 \Rightarrow a = 6q + 2$ **which is even.**

When $r = 3 \Rightarrow a = 6q + 3$ **which is odd.**

When $r = 4 \Rightarrow a = 6q + 4$ **which is even.**

When $r = 5 \Rightarrow a = 6q + 5$ **which is odd.**

This verifies that when r = 1 or 3 or 5; the integer obtained is $6q + 1$ or $6q + 3$ or $6q + 5$ and each of these integers is a positive odd number.

Hence the required result.

Ex.13 Use Euclid's Division Algorithm to show that the square of any positive integer is either of the form 3m or 3m + 1 for some integer m.

Sol. Let **a** and **b** are two positive integers such that **a** is greater than **b**; then:

$a = bq + r$; where q and r are also positive integers and $0 \le r < b$

Taking $b = 3$, we get:

$a = 3q + r$; where $0 \le r < 3$

$\Rightarrow$ The value of positive integer **a** will be
$3q + 0$, $3q + 1$ or $3q + 2$

i.e., $3q$, $3q + 1$ or $3q + 2$.

Now we have to show that the squares of positive integers 3q, 3q + 1 and 3q + 2 can be expressed as 3m, or 3m + 1 for some integer m.

$\therefore$ **Square of 3q** $= (3q)^2$

$= 9q^2 = 3(3q^2) = 3m$; 3 where m is some integer.

Square of 3q + 1 $= (3q + 1)^2$

$= 9q^2 + 6q + 1$

$= 3(3q^2 + 2q) + 1 = 3m + 1$ for some integer m.

Square of 3q + 2 $= (3q + 2)^2$

$= 9q^2 + 12q + 4$

$= 9q^2 + 12q + 3 + 1$

$= 3(3q^2 + 4q + 1) + 1 =$ **$3m + 1$** for some integer m.

$\therefore$ The square of any positive integer is either of the form 3m or 3m + 1 for some integer m.

Hence the required result.

Ex.14 Use Euclid's Division Algorithm to show that the cube of any positive integer is either of the 9m, 9m + 1 or 9m + 8 for some integer m.

Sol. Let **a** and **b** be two positive integers such that **a** is greater than **b**; then:

a = bq + r; where q and r are positive integers and $0 \leq r < b$.

Taking b = 3, we get:

$a = 3q + r$; where $0 \leq r < 3$

$\Rightarrow$ Different values of integer **a** are

3q, 3q + 1 or 3q + 2.

Cube of 3q = $(3q)^3 = 27q^3 = 9(3q^3) = 9m$; where **m** is some integer.

Cube of 3q + 1 = $(3q + 1)^3$

$= (3q)^3 + 3(3q)^2 \times 1 + 3(3q) \times 1^2 + 1^3$

$[\because (q + b)^3 = a^3 + 3a^2b + 3ab^2 + 1]$

$= 27q^3 + 27q^2 + 9q + 1$

$= 9(3q^3 + 3q^2 + q) + 1$

= **9m + 1**; where m is some integer.

Cube of 3q + 2 = $(3q + 2)^3$

$= (3q)^3 + 3(3q)^2 \times 2 + 3 \times 3q \times 2^2 + 2^3$

$= 27q^3 + 54q^2 + 36q + 8$

$= 9(3q^3 + 6q^2 + 4q) + 8$

= **9m + 8**; where m is some integer.

$\therefore$ Cube of any positive integer is of the form 9m or 9m + 1 or 9m + 8.

Hence the required result.

➤ THE FUNDAMENTAL THEOREM OF ARITHMETIC

Statement : Every composite number can be decomposed as a product prime numbers in a unique way, except for the order in which the prime numbers occur.

For example :

(i) $30 = 2 \times 3 \times 5, 30 = 3 \times 2 \times 5, 30 = 2 \times 5 \times 3$ and so on.

(ii) $432 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^4 \times 3^3$

or $432 = 3^3 \times 2^4$.

(iii) $12600 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7$

$= 2^3 \times 3^2 \times 5^2 \times 7$

In general, a composite number is expressed as the product of its prime factors written in ascending order of their values.

e.g., (i) $6615 = 3 \times 3 \times 3 \times 5 \times 7 \times 7$

$= 3^3 \times 5 \times 7^2$

(ii) $532400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 11 \times 11 \times 11$

❖EXAMPLES ❖

Ex.15 Consider the number 6^n, where n is a natural number. Check whether there is any value of $n \in N$ for which 6^n is divisible by 7.

Sol. Since, $6 = 2 \times 3$; $6^n = 2^n \times 3^n$

$\Rightarrow$ The prime factorisation of given number 6^n

$\Rightarrow$ **6^n is not divisible by 7.** **(Ans)**

Ex.16 Consider the number 12^n, where n is a natural number. Check whether there is any value of $n \in N$ for which 12^n ends with the digit zero.

Sol. We know, if any number ends with the digit zero it is always divisible by 5.

$\Rightarrow$ If 12^n ends with the digit zero, it must be divisible by 5.

This is possible only if prime factorisation of 12^n contains the prime number 5.

Now, $12 = 2 \times 2 \times 3 = 2^2 \times 3$

$\Rightarrow$ $12^n = (2^2 \times 3)^n = 2^{2n} \times 3^n$

i.e., prime factorisation of 12^n does not contain the prime number 5.

$\Rightarrow$ **There is no value of $n \in N$ for which**

12^n ends with the digit zero. **(Ans)**

➤ USING THE FACTOR TREE

❖EXAMPLES ❖

Ex.17 Find the prime factors of :

(i) 540 (ii) 21252 (iii) 8232

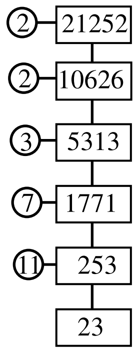
(i)

2	540	
2	270	540 divided by 2 gives 270
3	135	270 divided by 2 gives 135
3	45	135 divided by 3 gives 45
3	15	45 divided by 3 gives 15
	5	15 divided by 3 gives 5

5 is a prime number and so cannot be further divided by any prime number

$\therefore$ $540 = 2 \times 2 \times 3 \times 3 \times 3 \times 5 = 2^2 \times 3^3 \times 5$

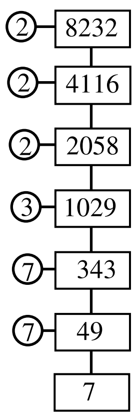
(ii)



$\therefore \quad 21252 = 2 \times 2 \times 3 \times 7 \times 11 \times 23$

$= \mathbf{2^2 \times 3 \times 11 \times 7 \times 23.}$

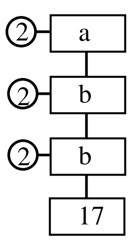
(iii)



$\therefore \quad 8232 = 2 \times 2 \times 2 \times 3 \times 7 \times 7 \times 7$

$= \mathbf{2^3 \times 3 \times 7^3.}$

Ex.18 Find the missing numbers a, b and c in the following factorisation:



Can you find the number on top without finding the other ?

Sol. c = 17 × 2 = 34

b = c × 2 = 34 × 2 = 68 and

a = b × 2 = 68 × 2 = 136

i.e., a = 136, b = 68 and c = 34. **(Ans)**

Yes, we can find the number on top without finding the others.

Reason: The given numbers 2, 2, 2 and 17 are the only prime factors of the number on top and so the number on top = 2 × 2 × 2 × 17 = 136

➤ USING THE FUNDAMENTAL THEOREM OF ARITHMETIC TO FIND H.C.F. AND L.C.M.

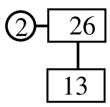
❖EXAMPLES ❖

Ex.19 Find the L.C.M. and H.C.F. of the following pairs of integers by applying the Fundamental theorem of Arithmetic method i.e., using the prime factorisation method.

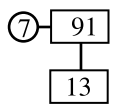
(i) 26 and 91 (ii) 1296 and 2520

(iii) 17 and 25

Sol. (i) Since, $26 = 2 \times 13$ and, $91 = 7 \times 13$



and



$\therefore$ **L.C.M.** = Product of each prime factor with highest powers. $= 2 \times 13 \times 7 = 182$. **(Ans)**

i.e., **L.C.M.** (26, 91) = 182. **(Ans)**

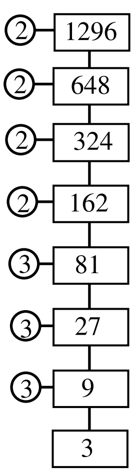
H.C.F. = Product of common prime factors with lowest powers. = 13.

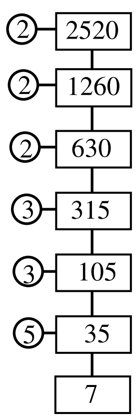
i.e., **H.C.F** (26, 91) = 13.

(ii) Since, $1296 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3 = 2^4 \times 3^4$

and, $2520 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7$

$= 2^3 \times 3^2 \times 5 \times 7$





$\therefore$ **L.C.M.** = Product of each prime factor with highest powers

$= 2^4 \times 3^4 \times 5 \times 7 =$ **45,360**

i.e., **L.C.M.** (1296, 2520) = 45,360 **(Ans)**

H.C.F. = Product of common prime factors with lowest powers.

$= 2^3 \times 3^2 = 8 \times 9 = 72$

i.e., H.C.F. (1296, 2520) = **72.** **(Ans)**

(iii) Since, $17 = 17$

and, $25 = 5 \times 5 = 5^2$

$\therefore$ **L.C.M.** $= 17 \times 5^2 = 17 \times 25 = 425$

and, **H.C.F.** = Product of common prime factors with lowest powers

= 1, as given numbers do not have any common prime factor.

In example 19 (i) :

Product of given two numbers $= 26 \times 91 = 2366$

and, product of their

L.C.M. and **H.C.F.** $= 182 \times 13 = 2366$

$\therefore$ Product of L.C.M and H.C.F of two given numbers = Product of the given numbers

In example 19 (ii) :

Product of given two numbers

$= 1296 \times 2520 = 3265920$

and, product of their

L.C.M. and **H.C.F.** $= 45360 \times 72 = 3265920$

$\therefore$ L.C.M. (1296, 2520) × H.C.F. (1296, 2520)

$= 1296 \times 2520$

In example 19 (iii) :

The given numbers 17 and 25 do not have any common prime factor. Such numbers are called co-prime numbers and their H.C.F. is always equal to 1 (one), whereas their L.C.M. is equal to the product of the numbers.

But in case of two co-prime numbers also, the product of the numbers is always equal to the product of their L.C.M. and their H.C.F.

As, in case of co-prime numbers 17 and 25;

H.C.F. = 1; L.C.M. $= 17 \times 25 = 425$;

product of numbers $= 17 \times 25 = 425$

and product of their H.C.F. and L.C.M.

$= 1 \times 425 = 425$.

> - For any two positive integers :
>   Their L.C.M. × their H.C.F.
>   = Product of the number
>
> $\Rightarrow$ (i) L.C.M. $= \dfrac{\text{Product of the numbers}}{\text{H.C.F.}}$
>
>   (ii) H.C.F. $= \dfrac{\text{Product of the numbers}}{\text{L.C.M.}}$
>
>   (iii) One number $= \dfrac{\text{L.C.M.} \times \text{H.C.F.}}{\text{Other number}}$

Ex.20 Given that H.C.F. (306, 657) = 9,

find L.C.M. (306, 657)

Sol. H.C.F. (306, 657) = 9 means H.C.F. of

306 and 657 = 9

Required L.C.M. (306, 657) means required L.C.M. of 306 and 657.

For any two positive integers;

their L.C.M. = $\frac{\text{Product of the numbers}}{\text{Their H.C.F.}}$

i.e., L.C.M. (306, 657) = $\frac{306 \times 657}{9}$ = 22,338.

Ex.21 Given that L.C.M. (150, 100) = 300, find H.C.F. (150, 100)

Sol. L.C.M. (150, 100) = 300

$\Rightarrow$ L.C.M. of 150 and 100 = 300

Since, the product of number 150 and 100

= 150 × 100

And, we know :

H.C.F. (150, 100) = $\frac{\text{Product of 150 and 100}}{\text{L.C.M.}(150, 100)}$

$= \frac{150 \times 100}{300} = 50.$

Ex.22 The H.C.F. and L.C.M. of two numbers are 12 and 240 respectively. If one of these numbers is 48; find the other numbers.

Sol. Since, the product of two numbers

= Their H.C.F. × Their L.C.M.

$\Rightarrow$ One no. × other no. = H.C.F. × L.C.M.

$\Rightarrow$ Other no. = $\frac{12 \times 240}{48}$ = 60.

23 Explain why 7 × 11 × 13 + 13 and

7 × 6 × 5 × 4 × 3 + 5 are composite numbers.

Sol. Since,

7 × 11 × 13 + 13 = 13 × (7 × 11 + 1)

= 13 × 78 = 13 × 13 × 3 × 2;

that is, the given number has more than two factors and it is a composite number.

Similarly, 7 × 6 × 5 × 4 × 3 + 5

= 5 × (7 × 6 × 4 × 3 + 1)

= 5 × 505 = 5 × 5 × 101

$\Rightarrow$ The given no. is a composite number.

2
CHAPTER

POLYNOMIALS

CONTENTS

INTRODUCTION

Algebra is that branch of mathematics which treats the relation of numbers.

CONSTANTS AND VARIABLES

In algebra, two types of symbols are used: constants and variable (literals).

Constant :

It is a symbol whose value always remains the same, whatever the situation be.

For example: 5, –9, $\frac{3}{8}$, π, $\frac{7}{15}$, etc.

Variable :

It is a symbol whose value changes according to the situation.

For example : x, y, z, ax, a + x, 5y, – 7x, etc.

ALGEBRAIC EXPRESSION

(a) An algebraic expression is a collection of terms separated by plus (+) or minus (–) sign. For example : 3x + 5y, 7y – 2x, 2x – ay + az, etc.

(b) The various parts of an algebraic expression that are separated by '+' or '–' sign are called terms.

For example :

	Algebraic expression	**No. of terms**	**Terms**
(i)	$-32x$	1	$-32x$
(ii)	$2x + 3y$	2	$2x$ and $3y$
(iii)	$ax - 5y + cz$	3	ax, $-5y$ and cz
(iv)	$\frac{3}{x} + \frac{y}{7} - \frac{xy}{8} + 9$	4	$\frac{3}{x}, \frac{y}{7}, -\frac{xy}{8}$ and 9 & so on.

Types of Algebraic Expressions :

(i) Monomial : An algebraic expression having only one term is called a monomial. For ex. 8y, –7xy, $4x^2$, abx, etc. 'mono' means 'one'.

(ii) Binomial : An algebraic expression having two terms is called a binomial. For ex. 8x + 3y, 8x + 3, 8 + 3y, a + bz, 9 – 4y, $2x^2 - 4z$, $6y^2 - 5y$, etc. 'bi' means 'two'.

(iii) Trinomial : An algebraic expression having three terms is called a trinomial. For ex. ax – 5y + 8z, $3x^2 + 4x + 7$, $9y^2 - 3y + 2x$, etc. 'tri means 'three'.

(iv) Multinomial : An algebraic expression having two or more terms is called a multinomial.

➤ FACTORS AND COEFFICIENTS

◈ Factor :

Each combination of the constants and variables, which form a term, is called a factor.

For examples :

(i) 7, x and 7x are factors of 7x, in which 7 is constant (numerical) factor and x is variable (literal) factor.

(ii) In $\mathbf{-5x^2y}$, the numerical factor is -5 and literal factors are : x, y, xy, x^2 and x^2y.

◈ Coefficient :

Any factor of a term is called the coefficient of the remaining term.

For example :

(i) In **7x** ; 7 is coefficient of x

(ii) In $\mathbf{-5x^2y;}$ 5 is coefficient of $-x^2y$; -5 is coefficient of x^2y.

Ex. 1 Write the coefficient of :

(i) x^2 in $3x^3 - 5x^2 + 7$

(ii) xy in 8xyz

(iii) $-y$ in $2y^2 - 6y + 2$

(iv) x^0 in $3x + 7$

Sol. (i) -5

(ii) 8z

(iii) 6

(iv) Since $x^0 = 1$, Therefore

$3x + 7 = 3x + 7x^0$

coefficient of x^0 is 7.

➤ DEGREE OF A POLYNOMIAL

The greatest power (exponent) of the terms of a polynomial is called degree of the polynomial.

For example :

(a) In polynomial $5x^2 - 8x^7 + 3x$:

(i) The power of term $5x^2 = 2$

(ii) The power of term $-8x^7 = 7$

(iii) The power of $3x = 1$

Since, the greatest power is 7, therefore degree of the polynomial $5x^2 - 8x^7 + 3x$ is 7

(b) The degree of polynomial :

(i) $4y^3 - 3y + 8$ is 3

(ii) $7p + 2$ is $1(p = p^1)$

(iii) $2m - 7m^8 + m^{13}$ is 13 and so on.

❖ EXAMPLES ❖

Ex.2 Find which of the following algebraic expression is a polynomial.

(i) $3x^2 - 5x$ (ii) $x + \frac{1}{x}$

(iii) $\sqrt{y} - 8$ (iv) $z^5 - \sqrt[3]{z} + 8$

Sol. (i) $3x^2 - 5x = 3x^2 - 5x^1$

It is a polynomial.

(ii) $x + \frac{1}{x} = x^1 + x^{-1}$

It is not a polynomial.

(iii) $\sqrt{y} - 8 = y^{1/2} - 8$

Since, the power of the first term ($\sqrt{y}$) is $\frac{1}{2}$, which is not a whole number.

(iv) $z^5 - \sqrt[3]{z} + 8 = z^5 - z^{1/3} + 8$

Since, the exponent of the second term is 1/3, which in not a whole number. Therefore, the given expression is not a polynomial.

Ex.3 Find the degree of the polynomial :

(i) $5x - 6x^3 + 8x^7 + 6x^2$

(ii) $2y^{12} + 3y^{10} - y^{15} + y + 3$

(iii) x

(iv) 8

Sol. (i) Since the term with highest exponent (power) is $8x^7$ and its power is 7.

$\therefore$ The degree of given polynomial is 7.

(ii) The highest power of the variable is 15

$\Rightarrow$ degree = 15.

(iii) $x = x^1 \Rightarrow$ degree is 1.

(iv) $8 = 8x^0 \Rightarrow$ degree = 0

TYPES OF POLYNOMIALS

(A) Based on degree :

If degree of polynomial is

			Examples
1.	One	Linear	$x + 3,\ y - x + 2,\ \sqrt{3}\ x - 3$
2.	Two	Quadratic	$2x^2 - 7, \frac{1}{3}x^2 + y^2 - 2xy, x^2 + 1 + 3y$
3.	Three	Cubic	$x^3 + 3x^2 - 7x + 8,\ 2x^2 + 5x^3 + 7,$
4.	Four	bi-quadratic	$x^4 + y^4 + 2x^2y^2,\ x^4 + 3, \ldots$

(B) Based on Terms :

If number of terms in polynomial is

			Examples
1.	One	Monomial	$7x,\ 5x^9,\ \frac{7}{3}x^{16},\ xy, \ldots\ldots$
2.	Two	Binomial	$2 + 7y^6,\ y^3 + x^{14},\ 7 + 5x^9, \ldots$
3.	Three	Trinomial	$x^3 - 2x + y,\ x^{31} + y^{32} + z^{33}, \ldots..$

Note : (1) Degree of constant polynomials (**Ex.**5, 7, –3, 8/5, …) is zero.

(2) Degree of zero polynomial (zero = 0 = zero polynomial) is not defined.

POLYNOMIAL IN ONE VARIABLE

If a polynomial has only one variable then it is called polynomial in one variable.

Ex. $P(x) = 2x^3 + 5x - 3$ Cubic trinomial

$Q(x) = 7x^7 - 5x^5 - 3x^3 + x + 3$ polynomial of degree 7

$R(y) = y$ Linear, monomial

$S(t) = t^2 + 3$ Quadratic Binomial

Note : General form of a polynomial in one variable x of degree 'n' is $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$, $a_n \neq 0$, where $a_n, a_{n-1}, \ldots a_2, a_1, a_0$ all are constants.

$\therefore$ for linear $ax + b,\quad a \neq 0$

for quadratic $ax^2 + bx + c,\quad a \neq 0$

for cubic $ax^3 + bx^2 + cx + d,\ a \neq 0$

REMAINDER THEOREM

(i) Remainder obtained on dividing polynomial p(x) by x – a is equal to p(a) .

(ii) If a polynomial p(x) is divided by (x + a) the remainder is the value of p(x) at x = –a.

(iii) (x – a) is a factor of polynomial p(x) if p(a) = 0

(iv) (x + a) is a factor of polynomial p(x) if p(–a) = 0

(v) (x – a)(x – b) is a factor of polynomial p(x), if p(a) = 0 and p(b) = 0.

❖ EXAMPLES ❖

Ex.4 Find the remainder when $4x^3 - 3x^2 + 2x - 4$ is divided by

(a) $x - 1$ (b) $x + 2$ (c) $x + \frac{1}{2}$

Sol. Let $p(x) = 4x^3 - 3x^2 + 2x - 4$

(a) When p(x) is divided by (x – 1), then by remainder theorem, the required remainder will be p(1)

$p(1) = 4(1)^3 - 3(1)^2 + 2(1) - 4$

$= 4 \times 1 - 3 \times 1 + 2 \times 1 - 4$

$= 4 - 3 + 2 - 4 = -1$

(b) When p(x) is divided by (x + 2), then by remainder theorem, the required remainder will be p (–2).

$p(-2) = 4(-2)^3 - 3(-2)^2 + 2(-2) - 4$

$= 4 \times (-8) - 3 \times 4 - 4 - 4$

$= -32 - 12 - 8 = -52$

(c) When p(x) is divided by, $\left(x + \frac{1}{2}\right)$ then by remainder theorem, the required remainder will be

$$p\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) - 4$$

$$= 4 \times \left(-\frac{1}{8}\right) - 3 \times \frac{1}{4} - 2 \times \frac{1}{2} - 4$$

$$= -\frac{1}{2} - \frac{3}{4} - 1 - 4 = \frac{1}{2} - \frac{3}{4} - 5$$

$$= \frac{-2 - 3 - 20}{4} = \frac{-25}{4}$$

VALUES OF A POLYNOMIAL

For a polynomial $f(x) = 3x^2 - 4x + 2$.

To find its value at $x = 3$;

replace x by 3 everywhere.

So, the value of $f(x) = 3x^2 - 4x + 2$ at $x = 3$ is

$f(3) = 3 \times 3^2 - 4 \times 3 + 2$

$= 27 - 12 + 2 = 17.$

Similarly, the value of polynomial

$f(x) = 3x^2 - 4x + 2,$

(i) at $x = -2$ is $f(-2) = 3(-2)^2 - 4(-2) + 2$

$= 12 + 8 + 2 = 22$

(ii) at $x = 0$ is $f(0) = 3(0)^2 - 4(0) + 2$

$= 0 - 0 + 2 = 2$

(iii) at $x = \frac{1}{2}$ is $f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 2$

$= \frac{3}{4} - 2 + 2 = \frac{3}{4}$

Ex.5 Find the value of the polynomial $5x - 4x^2 + 3$ at:

(i) $x = 0$ (ii) $x = -1$

Sol. Let $p(x) = 5x - 4x^2 + 3$.

(i) At $x = 0$, $p(0) = 5 \times 0 - 4 \times (0)^2 + 3$

$= 0 - 0 + 3 = 3$

(ii) At $x = -1$, $p(-1) = 5(-1) - 4(-1)^2 + 3$

$= -5 - 4 + 3 = -6$

ZEROES OF A POLYNOMIAL

If for $x = a$, the value of the polynomial $p(x)$ is 0 i.e., $p(a) = 0$; then $x = a$ is a zero of the polynomial $p(x)$.

For example :

(i) For polynomial $p(x) = x - 2$; $p(2) = 2 - 2 = 0$

$\therefore$ $x = 2$ or simply 2 is a zero of the polynomial

$p(x) = x - 2$.

(ii) For the polynomial $g(u) = u^2 - 5u + 6$;

$g(3) = (3)^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

$\therefore$ 3 is a zero of the polynomial $g(u)$

$= u^2 - 5u + 6$.

Also, $g(2) = (2)^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

$\therefore$ 2 is also a zero of the polynomial

$g(u) = u^2 - 5u + 6$

(a) Every linear polynomial has one and only one zero.

(b) A given polynomial may have more than one zeroes.

(c) If the degree of a polynomial is n; the largest number of zeroes it can have is also n.

For example :

If the degree of a polynomial is 5, the polynomial can have at the most 5 zeroes; if the degree of a polynomial is 8; largest number of zeroes it can have is 8.

(d) A zero of a polynomial need not be 0.

For example : If $f(x) = x^2 - 4$,

then $f(2) = (2)^2 - 4 = 4 - 4 = 0$

Here, zero of the polynomial $f(x) = x^2 - 4$ is 2 which itself is not 0.

(e) 0 may be a zero of a polynomial.

For example : If $f(x) = x^2 - x$,

then $f(0) = 0^2 - 0 = 0$

Here 0 is the zero of polynomial

$f(x) = x^2 - x$.

❖ EXAMPLES ❖

Ex.6 Verify whether the indicated numbers are zeroes of the polynomial corresponding to them in the following cases :

(i) $p(x) = 3x + 1$, $x = -\frac{1}{3}$

(ii) $p(x) = (x + 1)(x - 2)$, $x = -1, 2$

(iii) $p(x) = x^2$, $x = 0$

(iv) $p(x) = \ell x + m$, $x = -\frac{m}{\ell}$

(v) $p(x) = 2x + 1$, $x = \frac{1}{2}$

Sol. (i) $p(x) = 3x + 1$

$\Rightarrow p\left(-\frac{1}{3}\right) = 3 \times -\frac{1}{3} + 1 = -1 + 1 = 0$

$\therefore$ $x = -\frac{1}{3}$ is a zero of $p(x) = 3x + 1$.

(ii) $p(x) = (x + 1)(x - 2)$

$\Rightarrow$ $p(-1) = (-1 + 1)(-1 - 2) = 0 \times -3 = 0$

and, $p(2) = (2 + 1)(2 - 2) = 3 \times 0 = 0$

$\therefore$ $x = -1$ and $x = 2$ are zeroes of the given polynomial.

(iii) $p(x) = x^2$ $\Rightarrow$ $p(0) = 0^2 = 0$

$\therefore$ $x = 0$ is a zero of the given polynomial

(iv) $p(x) = \ell x + m \Rightarrow p\left(-\dfrac{m}{\ell}\right) = \ell\left(-\dfrac{m}{\ell}\right) + m$

$= -m + m = 0$

$\therefore$ $x = -\dfrac{m}{\ell}$ is a zero of the given polynomial.

(v) $p(x) = 2x + 1 \Rightarrow p\left(\dfrac{1}{2}\right) = 2 \times \dfrac{1}{2} + 1$

$= 1 + 1 = 2 \neq 0$

$\therefore$ $x = \dfrac{1}{2}$ is not a zero of the given polynomial.

Ex.7 Find the zero of the polynomial in each of the following cases :

(i) $p(x) = x + 5$ (ii) $p(x) = 2x + 5$

(iii) $p(x) = 3x - 2$

Sol. To find the zero of a polynomial $p(x)$ means to solve the polynomial equation $p(x) = 0$.

(i) For the zero of polynomial $p(x) = x + 5$

$p(x) = 0$ $\Rightarrow$ $x + 5 = 0$ $\Rightarrow$ $x = -5$

$\therefore$ $x = -5$ is a zero of the polynomial $p(x) = x + 5$.

(ii) $p(x) = 0$ $\Rightarrow$ $2x + 5 = 0$

$\Rightarrow$ $2x = -5$ and $x = \dfrac{-5}{2}$

$\therefore$ $x = \dfrac{-5}{2}$ is a zero of $p(x) = 2x + 5$.

(iii) $p(x) = 0 \Rightarrow$ $3x - 2 = 0$

$\Rightarrow$ $3x = 2$ and $x = \dfrac{2}{3}$.

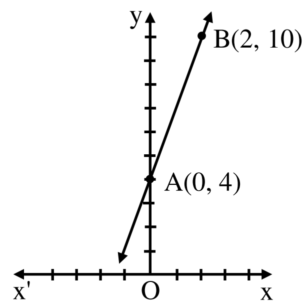
$\therefore$ $x = \dfrac{2}{3}$ is zero of $p(x) = 3x - 2$

GEOMETRIC MEANING OF THE ZEROES OF A POLYNOMIAL

Let us consider linear polynomial $ax + b$. The graph of $y = ax + b$ is a straight line.

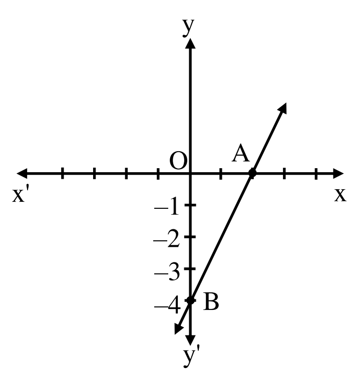
For example : The graph of $y = 3x + 4$ is a straight line passing through (0, 4) and (2, 10).

x	0	2
$y = 3x + 4$	4	10
Points	A	B



(i) Let us consider the graph of $y = 2x - 4$ intersects the x-axis at $x = 2$. The zero $2x - 4$ is 2. Thus, the zero of the polynomial $2x - 4$ is the x-coordinate of the point where the graph $y = 2x - 4$ intersects the x-axis.

x	2	0
$y = 2x - 4$	0	-4
Points	A	B



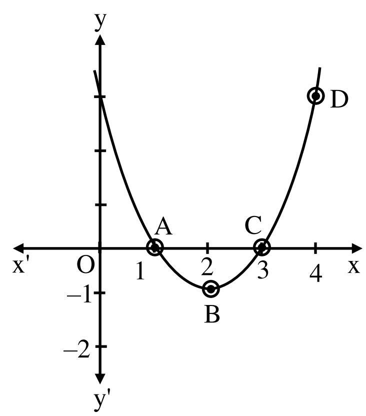
(ii) A general equation of a linear polynomial is $ax + b$. The graph of $y = ax + b$ is a straight line which intersects the x-axis at $\left(\dfrac{-b}{a}, 0\right)$.

Zero of the polynomial $ax + b$ is the x-coordinate of the point of intersection of the graph with x-axis.

(iii) Let us consider the quadratic polynomial $x^2 - 4x + 3$. The graph of $x^2 - 4x + 3$ intersects the x-axis at the point (1, 0) and (3, 0). Zeroes of the polynomial $x^2 - 4x + 3$ are the x-coordinates of the points of intersection of the graph with x-axis.

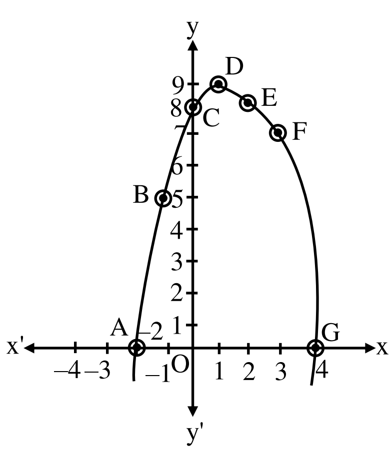
x	1	2	3	4	5
$y = x^2 - 4x + 3$	0	−1	0	3	8
Points	A	B	C	D	E

The shape of the graph of the quadratic polynomials is $\cup$ and the curve is known as parabola.



(iv) Now let us consider one more polynomial $-x^2 + 2x + 8$. Graph of this polynomial intersects the x-axis at the points (4, 0), (–2, 0). Zeroes of the polynomial $-x^2 + 2x + 8$ are the x-coordinates of the points at which the graph intersects the x-axis. The shape of the graph of the given quadratic polynomial is $\cap$ and the curve is known as parabola.

x	−2	−1	0	1	2	3	4
y	0	5	8	9	8	7	0
Points	A	B	C	D	E	F	G



The zeroes of a quadratic polynomial $ax^2 + bx + c$ he x-coordinates of the points where the graph of $y = ax^2 + bx + c$ intersects the x-axis.

Cubic polynomial : Let us find out geometrically how many zeroes a cubic has.

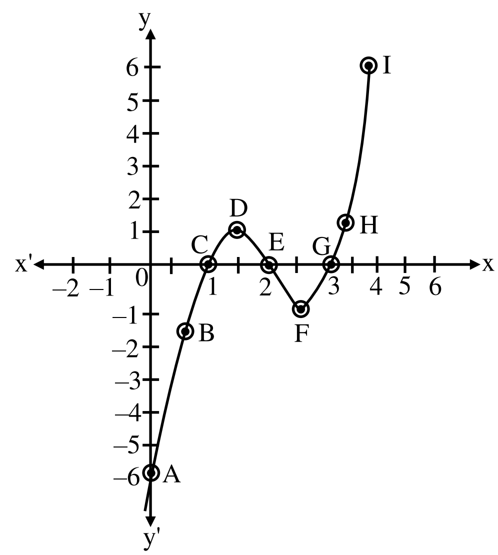
Let consider cubic polynomial

$x^3 - 6x^2 + 11x - 6$.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$y = x^3 - 6x^2 + 11x - 6$	−6	−1.875	0	0.375	0	−0.375	0	1.875	6
Points	A	B	C	D	E	F	G	H	I

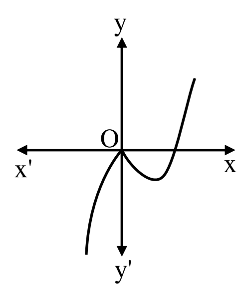
Case 1 :

The graph of the cubic equation intersects the x-axis at three points (1, 0), (2, 0) and (3, 0). Zeroes of the given polynomial are the x-coordinates of the points of intersection with the x-axis.



Case 2 :

The cubic equation $x^3 - x^2$ intersects the x-axis at the point (0, 0) and (1, 0). Zero of a polynomial $x^3 - x^2$ are the x-coordinates of the point where the graph cuts the x-axis.

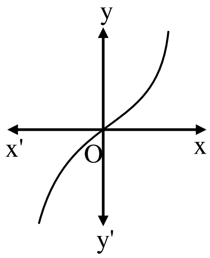


Zeroes of the cubic polynomial are 0 and 1.

Case 3 :

$y = x^3$

Cubic polynomial has only one zero.

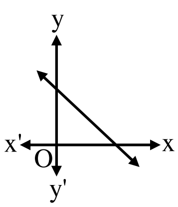


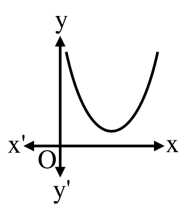
In brief : A cubic equation can have 1 or 2 or 3 zeroes or any polynomial of degree three can have at most three zeroes.

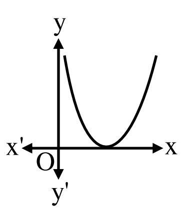
Remarks : In general, polynomial of degree n, the graph of y = p(x) passes x-axis at most at n points. Therefore, a polynomial p(x) of degree n has at most n zeroes.

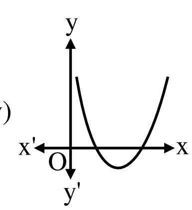
❖ EXAMPLES ❖

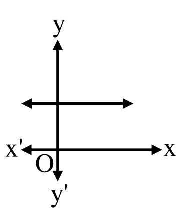
Ex.8 Which of the following correspond to the graph to a linear or a quadratic polynomial and find the number of zeroes of polynomial.

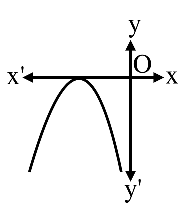
(i)


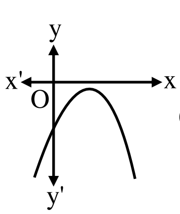
(ii)


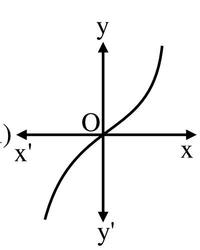
(iii)


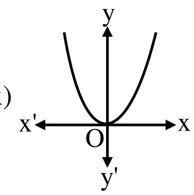
(iv)


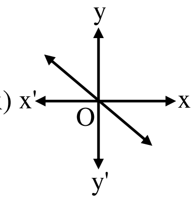
(v)


(vi)


(vii)


(viii)


(ix)


(x)


Sol. (i) The graph is a straight line so the graph is of a linear polynomial. The number of zeroes is one as the graph intersects the x-axis at one point only.

(ii) The graph is a parabola. So, this is the graph of quadratic polynomial. The number of zeroes is zero as the graph does not intersect the x-axis.

(iii) Here the polynomial is quadratic as the graph is a parabola. The number of zeroes is one as the graph intersects the x-axis at one point only (two coincident points).

(iv) Here, the polynomial is quadratic as the graph is a parabola. The number of zeroes is two as the graph intersects the x-axis at two points.

(v) The polynomial is linear as the graph is straight line. The number of zeroes is zero as the graph does not intersect the x-axis.

(vi) The polynomial is quadratic as the graph is a parabola. The number of zeroes is 1 as the graph intersects the x-axis at one point (two coincident points) only.

(vii) The polynomial is quadratic as the graph is a parabola. The number of zeroes is zero, as the graph does not intersect the x-axis.

(viii) Polynomial is neither linear nor quadratic as the graph is neither a straight line nor a parabola is one as the graph intersects the x-axis at one point only.

(ix) Here, the polynomial is quadratic as the graph is a parabola. The number of zeroes is one as the graph intersects the x-axis at one point only (two coincident points).

(x) The polynomial is linear as the graph is a straight line. The number of zeroes is one as the graph intersects the x-axis at only one point.

➤ RELATIONSHIP BETWEEN THE ZEROES AND THE COEFFICIENTS OF A POLYNOMIAL.

Consider quadratic polynomial

$P(x) = 2x^2 - 16x + 30.$

Now, $2x^2 - 16x + 30 = (2x - 6)(x - 3)$

$= 2(x - 3)(x - 5)$

The zeroes of P(x) are 3 and 5.

Sum of the zeroes

$$= 3 + 5 = 8 = \frac{-(-16)}{2} = -\left[\frac{\text{coefficient of } x}{\text{coefficient of } x^2}\right]$$

Product of the zeroes

$$= 3 \times 5 = 15 = \frac{30}{2} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

So if $ax^2 + bx + c$, $a \neq 0$ is a quadratic polynomial and α, β are two zeroes of polynomial then $\boxed{\alpha + \beta = -\frac{b}{a}}$, $\boxed{\alpha\beta = \frac{c}{a}}$

❖ EXAMPLES ❖

Ex.9 Find the zeroes of the quadratic polynomial $6x^2 - 13x + 6$ and verify the relation between the zeroes and its coefficients.

Sol. We have, $6x^2 - 13x + 6 = 6x^2 - 4x - 9x + 6$

$= 2x(3x - 2) - 3(3x - 2)$

$= (3x - 2)(2x - 3)$

So, the value of $6x^2 - 13x + 6$ is 0, when $(3x - 2) = 0$ or $(2x - 3) = 0$ i.e.,

When $x = \frac{2}{3}$ or $\frac{3}{2}$

Therefore, the zeroes of $6x^2 - 13x + 6$ are

$\frac{2}{3}$ and $\frac{3}{2}$.

Sum of the zeroes

$$= \frac{2}{3} + \frac{3}{2} = \frac{13}{6} = \frac{-(-13)}{6} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= \frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.10 Find the zeroes of the quadratic polynomial $4x^2 - 9$ and verify the relation between the zeroes and its coefficients.

Sol. We have,

$4x^2 - 9 = (2x)^2 - 3^2 = (2x - 3)(2x + 3)$

So, the value of $4x^2 - 9$ is 0, when

$2x - 3 = 0$ or $2x + 3 = 0$

i.e., when $x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

Therefore, the zeroes of $4x^2 - 9$ are $\frac{3}{2}$ & $-\frac{3}{2}$.

Sum of the zeroes

$$= \frac{3}{2} - \frac{3}{2} = 0 = \frac{-(0)}{4} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= \left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) = \frac{-9}{4} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.11 Find the zeroes of the quadratic polynomial $9x^2 - 5$ and verify the relation between the zeroes and its coefficients.

Sol. We have,

$9x^2 - 5 = (3x)^2 - (\sqrt{5})^2 = (3x - \sqrt{5})(3x + \sqrt{5})$

So, the value of $9x^2 - 5$ is 0,

when $3x - \sqrt{5} = 0$ or $3x + \sqrt{5} = 0$

i.e., when $x = \frac{\sqrt{5}}{3}$ or $x = \frac{-\sqrt{5}}{3}$.

Sum of the zeroes

$$= \frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3} = 0 = \frac{-(0)}{9} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= \left(\frac{\sqrt{5}}{3}\right)\left(\frac{-\sqrt{5}}{3}\right) = \frac{-5}{9} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.12 If α and β are the zeroes of $ax^2 + bx + c$, $a \neq 0$ then verify the relation between the zeroes and its coefficients.

Sol. Since α and β are the zeroes of polynomial $ax^2 + bx + c$.

Therefore, $(x - \alpha)$, $(x - \beta)$ are the factors of the polynomial $ax^2 + bx + c$.

$\Rightarrow ax^2 + bx + c = k(x - \alpha)(x - \beta)$

$\Rightarrow ax^2 + bx + c = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$

$\Rightarrow ax^2 + bx + c = kx^2 - k(\alpha + \beta)x + k\alpha\beta$ ...(1)

Comparing the coefficients of x^2, x and constant terms of (1) on both sides, we get

$a = k, b = - k (\alpha + \beta)$ and $c = k\alpha\beta$

$\Rightarrow \quad \alpha + \beta = -\frac{b}{k}$ and $\alpha\beta = \frac{c}{k}$

$\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$ $[\because k = a]$

Sum of the zeroes $= \frac{-b}{a} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$

Product of the zeroes $= \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Ex. 13 Prove relation between the zeroes and the coefficient of the quadratic polynomial $ax^2 + bx + c$.

Sol. Let α and β be the zeroes of the polynomial $ax^2 + bx + c$

$$\therefore \quad \alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad ....(1)$$

$$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a} \qquad ....(2)$$

By adding (1) and (2), we get

$$\alpha + \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2b}{2a} = -\frac{b}{a} = \frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

Hence, sum of the zeroes of the polynomial $ax^2 + bx + c$ is $-\frac{b}{a}$

By multiplying (1) and (2), we get

$$\alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2-\sqrt{(b^2-4ac)^2}}{4a^2} = \frac{b^2-b^2+4ac}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

$$= \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Hence, product of zeroes $= \frac{c}{a}$

In general, it can be proved that if α, β, γ are the zeroes of a cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

Note, $\frac{b}{a}, \frac{c}{a}$ and $\frac{d}{a}$ are meaningful because $a \neq 0$.

Ex.14 find the zeroes of the quadratic polynomial $x^2 - 2x - 8$ and verify a relationship between zeroes and its coefficients.

Sol. $x^2 - 2x - 8 = x^2 - 4x + 2x - 8$

$= x (x - 4) + 2 (x - 4) = (x - 4) (x + 2)$

So, the value of $x^2 - 2x - 8$ is zero when $x - 4 = 0$ or $x + 2 = 0$ i.e., when $x = 4$ or $x = - 2$.

So, the zeroes of $x^2 - 2x - 8$ are $4, - 2$.

Sum of the zeroes

$$= 4 - 2 = 2 = \frac{-(-2)}{1} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= 4 (-2) = -8 = \frac{-8}{1} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.15 Verify that the numbers given along side of the cubic polynomials are their zeroes. Also verify the relationship between the zeroes and the coefficients. $2x^3 + x^2 - 5x + 2$; $\frac{1}{2}, 1, - 2$

Sol. Here, the polynomial p(x) is

$2x^3 + x^2 - 5x + 2$

Value of the polynomial $2x^3 + x^2 - 5x + 2$ when $x = 1/2$

$$= 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2 = \frac{1}{4} + \frac{1}{4} - \frac{5}{2} + 2 = 0$$

So, 1/2 is a zero of p(x).

On putting $x = 1$ in the cubic polynomial

$2x^3 + x^2 - 5x + 2$

$= 2(1)^3 + (1)^2 - 5(1) + 2 = 2 + 1 - 5 + 2 = 0$

On putting x = – 2 in the cubic polynomial

$2x^3 + x^2 - 5x + 2$

$= 2(-2)^3 + (-2)^2 - 5(-2) + 2$

$= -16 + 4 + 10 + 2 = 0$

Hence, $\frac{1}{2}$, 1, – 2 are the zeroes of the given polynomial.

Sum of the zeroes of p(x)

$$= \frac{1}{2} + 1 - 2 = -\frac{1}{2} = \frac{-\text{coefficient of } x^2}{\text{coefficient of } x^3}$$

Sum of the products of two zeroes taken at a time

$$= \frac{1}{2} \times 1 + \frac{1}{2} \times (-2) + 1 \times (-2)$$

$$= \frac{1}{2} - 1 - 2 = -\frac{5}{2} = \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$$

Product of all the three zeroes

$$= \left(\frac{1}{2}\right) \times (1) \times (-2) = -1$$

$$= \frac{-(2)}{2} = \frac{-\text{constant term}}{\text{coefficient of } x^3}$$

➤ SYMMETRIC FUNCTIONS OF ZEROS OF A QUADRATIC POLYNOMIAL.

◈ Symmetric function :

An algebraic expression in α and β, which remains unchanged, when α and β are interchanged is known as symmetric function in α and β.

For example, $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ etc. are symmetric functions. Symmetric function is to be expressed in terms of (α + β) and αβ. So, this can be evaluated for a given quadratic equation.

◈ Some useful relations involving α and β :

1. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
2. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$
3. $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta) = (\alpha + \beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$
4. $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
5. $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$
6. $\alpha^4 + \beta^4 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$
7. $\alpha^4 - \beta^4 = (\alpha^2 + \beta^2)(\alpha^2 - \beta^2)$ then use (1) and (3)

❖ EXAMPLES ❖

Ex.16 If α and β are the zeroes of the polynomial $ax^2 + bx + c$. Find the value of

(i) α – β (ii) $\alpha^2 + \beta^2$.

Sol. Since α and β are the zeroes of the polynomial $ax^2 + bx + c$.

$$\therefore \quad \alpha + \beta = -\frac{b}{a}; \quad \alpha\beta = \frac{c}{a}$$

(i) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - \frac{4c}{a} = \frac{b^2}{a^2} - \frac{4c}{a} = \frac{b^2 - 4ac}{a^2}$$

$$\alpha - \beta = \frac{\sqrt{b^2 - 4ac}}{a}$$

(ii) $\alpha^2 + \beta^2 = \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta$

$= (\alpha+\beta)^2 - 2\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2 - 2ac}{a^2}$$

Ex.17 If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$. Find the value of

(i) $\alpha^2 - \beta^2$ (ii) $\alpha^3 + \beta^3$.

Sol. Since α and β are the zeroes of $ax^2 + bx + c$

$$\therefore \quad \alpha + \beta = \frac{-b}{a}, \quad \alpha\beta = \frac{c}{a}$$

(i) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

$$= -\frac{b}{a}\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$

$$= -\frac{b}{a}\sqrt{\left(\frac{-b}{a}\right)^2 - 4\frac{c}{a}} = -\frac{b}{a}\sqrt{\frac{b^2 - 4ac}{a^2}}$$

$$= -\frac{b\sqrt{b^2 - 4ac}}{a^2}$$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$

$= (\alpha + \beta)[(\alpha^2 + \beta^2 + 2\alpha\beta) - 3\alpha\beta]$

$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$

$$= \frac{-b}{a}\left[\left(\frac{-b}{a}\right)^2 - \frac{3c}{a}\right]$$

$$= \frac{-b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right] = \frac{-b}{a}\left(\frac{b^2 - 3ac}{a^2}\right)$$

$$= \frac{-b^3 + 3abc}{a^3}$$

➢ TO FORM A QUADRATIC POLYNOMIAL WITH THE GIVEN ZEROES

Let zeroes of a quadratic polynomial be α and β.

$\therefore \quad x = \alpha, \quad x = \beta$

$x - \alpha = 0, \quad x - \beta = 0$

The obviously the quadratic polynomial is

$(x - \alpha)(x - \beta)$

i.e., $\quad x^2 - (\alpha + \beta)x + \alpha\beta$

$\boxed{x^2 - (\text{Sum of the zeroes})\,x + \text{Product of the zeroes}}$

❖ EXAMPLES ❖

Ex.18 Form the quadratic polynomial whose zeroes are 4 and 6.

Sol. Sum of the zeroes = 4 + 6 = 10

Product of the zeroes = 4 × 6 = 24

Hence the polynomial formed

$= x^2 -$ (sum of zeroes) x + Product of zeroes

$= x^2 - 10x + 24$

Ex.19 Form the quadratic polynomial whose zeroes are –3, 5.

Sol. Here, zeroes are – 3 and 5.

Sum of the zeroes = – 3 + 5 = 2

Product of the zeroes = (–3) × 5 = – 15

Hence the polynomial formed

$= x^2 -$ (sum of zeroes) x + Product of zeroes

$= x^2 - 2x - 15$

Ex.20 Find a quadratic polynomial whose sum of zeroes and product of zeroes are respectively-

(i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

Sol. Let the polynomial be $ax^2 + bx + c$ and its zeroes be α and β.

(i) Here, $\alpha + \beta = \frac{1}{4}$ and $\alpha \,.\, \beta = -1$

Thus the polynomial formed

$= x^2 -$ (Sum of zeroes) x + Product of zeroes

$= x^2 - \left(\frac{1}{4}\right)x - 1 = x^2 - \frac{x}{4} - 1$

The other polynomial are $k\left(x^2 - \frac{x}{4} - 1\right)$

If k = 4, then the polynomial is $4x^2 - x - 4$.

(ii) Here, $\alpha + \beta = \sqrt{2}$, $\alpha\beta = \frac{1}{3}$

Thus the polynomial formed

$= x^2 -$ (Sum of zeroes) x + Product of zeroes

$= x^2 - (\sqrt{2})\,x + \frac{1}{3}$ or $x^2 - \sqrt{2}\,x + \frac{1}{3}$

Other polynomial are $k\left(x^2 - \sqrt{2}x + \frac{1}{3}\right)$

If k = 3, then the polynomial is

$3x^2 - 3\sqrt{2}\,x + 1$

(iii) Here, $\alpha + \beta = 0$ and $\alpha.\beta = \sqrt{5}$

Thus the polynomial formed

$= x^2 -$ (Sum of zeroes) x + Product of zeroes

$= x^2 - (0)\,x + \sqrt{5} = x^2 + \sqrt{5}$

Ex.21 Find a cubic polynomial with the sum of its zeroes, sum of the products of its zeroes taken two at a time, and product of its zeroes as 2, – 7 and –14, respectively.

Sol. Let the cubic polynomial be

$ax^3 + bx^2 + cx + d$

$\Rightarrow \quad x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}$ ....(1)

and its zeroes are α, β and γ, then

$\alpha + \beta + \gamma = 2 = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \alpha\gamma = -7 = \frac{c}{a}$

$\alpha\beta\gamma = -14 = -\frac{d}{a}$

Putting the values of $\frac{b}{a}$, $\frac{c}{a}$ and $\frac{d}{a}$ in (1),

we get

$x^3 + (-2)\,x^2 + (-7)x + 14$

$\Rightarrow\ x^3 - 2x^2 - 7x + 14$

Ex.22 Find the cubic polynomial with the sum, sum of the product of its zeroes taken two at a time and product of its zeroes as 0, –7 and –6 respectively.

Sol. Let the cubic polynomial be

$ax^3 + bx^2 + cx + d$

$\Rightarrow\ x^3 + \frac{b}{a}x^2 + \frac{c}{a}\,x + \frac{d}{a}$ ....(1)

and its zeroes are α, β, γ. Then

$\alpha + \beta + \gamma = 0 = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \alpha\gamma = -7 = \frac{c}{a}$

$\alpha\beta\gamma = -6 = \frac{-d}{a}$

Putting the values of $\frac{b}{a}$, $\frac{c}{a}$ and $\frac{d}{a}$ in (1),

we get

$x^3 - (0)\,x^2 + (-7)\,x + (-6)$

or $x^3 - 7x + 6$

Ex.23 If α and β are the zeroes of the polynomials $ax^2 + bx + c$ then form the polynomial whose zeroes are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Sol. Since α and β are the zeroes of $ax^2 + bx + c$

So $\alpha + \beta = \frac{-b}{a}$, $\alpha\beta = \frac{c}{a}$

Sum of the zeroes $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$

$$= \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c}$$

Product of the zeroes

$$= \frac{1}{\alpha}\cdot\frac{1}{\beta} = \frac{1}{\frac{c}{a}} = \frac{a}{c}$$

But required polynomial is

x^2 – (sum of zeroes) x + Product of zeroes

$\Rightarrow\ x^2 - \left(\frac{-b}{c}\right)\,x + \left(\frac{a}{c}\right)$

or $x^2 + \frac{b}{c}\,x + \frac{a}{c}$

or $c\left(x^2 + \frac{b}{c}x + \frac{a}{c}\right)$

$\Rightarrow\ cx^2 + bx + a$

Ex.24 If α and β are the zeroes of the polynomial $x^2 + 4x + 3$, form the polynomial whose zeroes are $1 + \frac{\beta}{\alpha}$ and $1 + \frac{\alpha}{\beta}$.

Sol. Since α and β are the zeroes of the polynomial $x^2 + 4x + 3$.

Then, $\alpha + \beta = -4$, $\alpha\beta = 3$

Sum of the zeroes

$$= 1 + \frac{\beta}{\alpha} + 1 + \frac{\alpha}{\beta} = \frac{\alpha\beta + \beta^2 + \alpha\beta + \alpha^2}{\alpha\beta}$$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta}{\alpha\beta} = \frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{(-4)^2}{3} = \frac{16}{3}$$

Product of the zeroes

$$= \left(1 + \frac{\beta}{\alpha}\right)\left(1 + \frac{\alpha}{\beta}\right) = 1 + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha\beta}{\alpha\beta}$$

$$= 2 + \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{2\alpha\beta + \alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{(-4)^2}{3} = \frac{16}{3}$$

But required polynomial is

x^2 – (sum of zeroes) x + product of zeroes

or $x^2 - \frac{16}{3}\,x + \frac{16}{3}$ or $k\left(x^2 - \frac{16}{3}x + \frac{16}{3}\right)$

or $3\left(x^2 - \frac{16}{3}x + \frac{16}{3}\right)$ (if $k = 3$)

$\Rightarrow\ 3x^2 - 16x + 16$

➤ WORKING RULE TO DIVIDE A POLYNOMIAL BY ANOTHER POLYNOMIAL

Step 1:

First arrange the term of dividend and the divisor in the decreasing order of their degrees.

Step 2 :

To obtain the first term of quotient divide the highest degree term of the dividend by the highest degree term of the divisor.

Step 3 :

To obtain the second term of the quotient, divide the highest degree term of the new dividend obtained as remainder by the highest degree term of the divisor.

Step 4 :

Continue this process till the degree of remainder is less than the degree of divisor.

◈ Division Algorithm for Polynomial

If p(x) and g(x) are any two polynomials with

$g(x) \neq 0$, then we can find polynomials q(x) and r(x) such that

$p(x) = q(x) \times g(x) + r(x)$

where r(x) = 0 or degree of r(x) < degree of g(x).

The result is called Division Algorithm for polynomials.

$$\boxed{\textbf{Dividend} = \textbf{Quotient} \times \textbf{Divisor} + \textbf{Remainder}}$$

❖ EXAMPLES ❖

Ex.25 Divide $3x^3 + 16x^2 + 21x + 20$ by $x + 4$.

Sol.

$$\begin{array}{r|l}
 & 3x^2 + 4x + 5 \\ \hline
x+4 & 3x^3 + 16x^2 + 21x + 20 \\
 & 3x^3 + 12x^2 \\
 & - \quad - \\ \hline
 & \quad 4x^2 + 21x + 20 \\
 & \quad 4x^2 + 16x \\
 & \quad - \quad - \\ \hline
 & \qquad 5x + 20 \\
 & \qquad 5x + 20 \\
 & \qquad - \quad - \\ \hline
 & \qquad\quad 0
\end{array}$$

First term of q(x) $= \dfrac{3x^3}{x} = 3x^2$

Second term of q(x) $= \dfrac{4x^2}{x} = 4x$

Third term of q(x) $= \dfrac{5x}{x} = 5$

Quotient $= 3x^2 + 4x + 5$

Remainder = 0

Ex.26 Apply the division algorithm to find the quotient and remainder on dividing p(x) by g(x) as given below :

$p(x) = x^3 - 3x^2 + 5x - 3,\ g(x) = x^2 - 2$

Sol. We have,

$p(x) = x^3 - 3x^2 + 5x - 3$ and $g(x) = x^2 - 2$

$$\begin{array}{r|l}
 & x - 3 \\ \hline
x^2-2 & x^3 - 3x^2 + 5x - 3 \\
 & x^3 \qquad\quad - 2x \\
 & - \qquad\quad + \\ \hline
 & \quad - 3x^2 + 7x - 3 \\
 & \quad - 3x^2 \qquad + 6 \\
 & \quad + \qquad\quad - \\ \hline
 & \qquad\qquad 7x - 9
\end{array}$$

First term of quotient is $\dfrac{x^3}{x^2} = x$

Second term of quotient is $\dfrac{-3x^2}{x^2} = -3$

We stop here since

degree of $(7x - 9)$ < degree of $(x^2 - 2)$

So, quotient = x – 3, remainder = 7x – 9

Therefore,

Quotient × Divisor + Remainder

$= (x - 3)(x^2 - 2) + 7x - 9$

$= x^3 - 2x - 3x^2 + 6 + 7x - 9$

$= x^3 - 3x^2 + 5x - 3$ = Dividend

Therefore, the division algorithm is verified.

Ex.27 Apply the division algorithm to find the quotient and remainder on dividing p(x) by g(x) as given below

$p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

Sol. We have,

$p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

$$\begin{array}{r|l}
 & x^2 + x - 3 \\ \hline
x^2 - x + 1 & x^4 - 3x^2 + 4x + 5 \\
 & x^4 - x^3 + x^2 \\
 & - \quad + \quad - \\ \hline
 & \quad x^3 - 4x^2 + 4x + 5 \\
 & \quad x^3 - x^2 + x \\
 & \quad - \quad + \quad - \\ \hline
 & \qquad -3x^2 + 3x + 5 \\
 & \qquad -3x^2 + 3x - 3 \\
 & \qquad + \quad - \quad + \\ \hline
 & \qquad\qquad\quad 8
\end{array}$$

We stop here since

degree of (8) < degree of $(x^2 - x + 1)$.

So, quotient $= x^2 + x - 3$, remainder $= 8$

Therefore,

Quotient × Divisor + Remainder

$= (x^2 + x - 3)(x^2 - x + 1) + 8$

$= x^4 - x^3 + x^2 + x^3 - x^2 + x - 3x^2 + 3x - 3 + 8$

$= x^4 - 3x^2 + 4x + 5 \quad$ = Dividend

Therefore the Division Algorithm is verified.

Ex.28 Check whether the first polynomial is a factor of the second polynomial by applying the division algorithm. $t^2 - 3$; $2t^4 + 3t^3 - 2t^2 - 9t - 12$.

Sol. We divide $2t^4 + 3t^3 - 2t^2 - 9t - 12$ by $t^2 - 3$

$$
\begin{array}{r|l}
 & 2t^2 + 3t + 4 \\
\hline
t^2 - 3 & 2t^4 + 3t^3 - 2t^2 - 9t - 12 \\
 & 2t^4 \qquad - 6t^2 \\
 & - \qquad\quad + \\
\hline
 & 3t^3 + 4t^2 + 9t - 12 \\
 & 3t^3 \qquad - 9t \\
 & - \qquad\quad + \\
\hline
 & 4t^2 \qquad - 12 \\
 & 4t^2 \qquad - 12 \\
 & - \qquad\quad + \\
\hline
 & 0
\end{array}
$$

Here, remainder is 0, so $t^2 - 3$ is a factor of $2t^4 + 3t^3 - 2t^2 - 9t - 12$.

$2t^4 + 3t^3 - 2t^2 - 9t - 12 = (2t^2 + 3t + 4)(t^2 - 3)$

Ex.29 Obtain all the zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

Sol. Since two zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$,

$x = \sqrt{\frac{5}{3}}$, $x = -\sqrt{\frac{5}{3}}$

$\Rightarrow \left(x - \sqrt{\frac{5}{3}}\right)\left(x + \sqrt{\frac{5}{3}}\right) = x^2 - \frac{5}{3}$ or $3x^2 - 5$ is a factor of the given polynomial.

Now, we apply the division algorithm to the given polynomial and $3x^2 - 5$.

$$
\begin{array}{r|l}
 & x^2 + 2x + 1 \\
\hline
3x^2 - 5 & 3x^4 + 6x^3 - 2x^2 - 10x - 5 \\
 & 3x^4 \qquad - 5x^2 \\
 & - \qquad\quad + \\
\hline
 & 6x^3 + 3x^2 - 10x - 5 \\
 & 6x^3 \qquad - 10x \\
 & - \qquad\quad + \\
\hline
 & 3x^2 \qquad - 5 \\
 & 3x^2 \qquad - 5 \\
 & - \qquad\quad + \\
\hline
 & 0
\end{array}
$$

So, $3x^4 + 6x^3 - 2x^2 - 10x - 5$

$= (3x^2 - 5)(x^2 + 2x + 1) + 0$

Quotient $= x^2 + 2x + 1 = (x + 1)^2$

Zeroes of $(x + 1)^2$ are $-1, -1$.

Hence, all its zeroes are $\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}, -1, -1$.

Ex.30 On dividing $x^3 - 3x^2 + x + 2$ by a polynomial $g(x)$, the quotient and remainder were $x - 2$ and $-2x + 4$, respectively. Find $g(x)$.

Sol. $p(x) = x^3 - 3x^2 + x + 2$

$q(x) = x - 2$ and $r(x) = -2x + 4$

By Division Algorithm, we know that

$p(x) = q(x) \times g(x) + r(x)$

Therefore,

$x^3 - 3x^2 + x + 2 = (x - 2) \times g(x) + (-2x + 4)$

$\Rightarrow x^3 - 3x^2 + x + 2 + 2x - 4 = (x - 2) \times g(x)$

$\Rightarrow g(x) = \dfrac{x^3 - 3x^2 + 3x - 2}{x - 2}$

On dividing $x^3 - 3x^2 + 3x - 2$ by $x - 2$, we get $g(x)$

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x-2 & x^3 - 3x^2 + 3x - 2 \\
 & x^3 - 2x^2 \\
 & - \quad + \\
\hline
 & -x^2 + 3x - 2 \\
 & -x^2 + 2x \\
 & + \quad - \\
\hline
 & x - 2 \\
 & x - 2 \\
 & - \quad + \\
\hline
 & 0
\end{array}
$$

First term of quotient is $\dfrac{x^3}{x} = x$

Second term of quotient is $\dfrac{-x^2}{x} = -x$

Third term of quotient is $\dfrac{x}{x} = 1$

Hence, $g(x) = x^2 - x + 1$.

Ex.31 Give examples of polynomials p(x), q(x) and r(x), which satisfy the division algorithm and

(i) deg p(x) = deg q(x)

(ii) deg q(x) = deg r(x)

(iii) deg q(x) = 0

Sol. (i) Let $q(x) = 3x^2 + 2x + 6$, degree of q(x) = 2

$p(x) = 12x^2 + 8x + 24$, degree of p(x) = 2

Here, deg p(x) = deg q(x)

(ii) $p(x) = x^5 + 2x^4 + 3x^3 + 5x^2 + 2$

$q(x) = x^2 + x + 1$, degree of q(x) = 2

$g(x) = x^3 + x^2 + x + 1$

$r(x) = 2x^2 - 2x + 1$, degree of r(x) = 2

Here, deg q(x) = deg r(x)

(iii) Let $p(x) = 2x^4 + 8x^3 + 6x^2 + 4x + 12$

q(x) = 2, degree of q(x) = 0

$g(x) = x^4 + 4x^3 + 3x^2 + 2x + 6$

r(x) = 0

Here, deg q(x) = 0

Ex.32 If the zeroes of polynomial $x^3 - 3x^2 + x + 1$ are a – b, a , a + b. Find a and b.

Sol. $\because$ a – b, a, a + b are zeros

$\therefore$ product $(a - b)\,a(a + b) = -1$

$\Rightarrow (a^2 - b^2)\,a = -1$ …(1)

and sum of zeroes is $(a - b) + a + (a + b) = 3$

$\Rightarrow 3a = 3 \Rightarrow a = 1$ …(2)

by (1) and (2)

$(1 - b^2)1 = -1$

$\Rightarrow 2 = b^2 \Rightarrow b = \pm\sqrt{2}$

$\therefore a = -1$ & $b = \pm\sqrt{2}$ **Ans.**

Ex.33 If two zeroes of the polynomial

$x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.

Sol. $\because$ $2 \pm \sqrt{3}$ are zeroes.

$\therefore$ $x = 2 \pm \sqrt{3}$

$\Rightarrow$ $x - 2 = \pm\sqrt{3}$ (squaring both sides)

$\Rightarrow$ $(x - 2)^2 = 3$ $\Rightarrow$ $x^2 + 4 - 4x - 3 = 0$

$\Rightarrow$ $x^2 - 4x + 1 = 0$, is a factor of given polynomial

$\therefore$ other factors

$$= \frac{x^4 - 6x^3 - 26x^2 + 138x - 35}{x^2 - 4x + 1}$$

$$\begin{array}{r|l} & x^2 - 2x - 35 \\ \hline x^2 - 4x + 1 & x^4 - 6x^3 - 26x^2 + 138x - 35 \\ & x^4 - 4x^3 + x^2 \\ & - \quad + \quad - \\ \hline & -2x^3 - 27x^2 + 138x - 35 \\ & -2x^3 + 8x^2 - 2x \\ & + \quad - \quad + \\ \hline & -35x^2 + 140x - 35 \\ & -35x^2 + 140x - 35 \\ & + \quad - \quad + \\ \hline & \times \end{array}$$

$\therefore$ other factors $= x^2 - 2x - 35$

$= x^2 - 7x + 5x - 35 = x(x - 7) + 5(x - 7)$

$= (x - 7)(x + 5)$

$\therefore$ other zeroes are $(x - 7) = 0 \Rightarrow x = 7$

$x + 5 = 0 \Rightarrow x = -5$ **Ans.**

Ex.34 If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be x + a, find k & a.

Sol.

$$\begin{array}{r|l} & x^2 - 4x + (8 - k) \\ \hline x^2 - 2x + k & x^4 - 6x^3 + 16x^2 - 25x + 10 \\ & x^4 - 2x^3 + x^2k \\ & - \quad + \quad - \\ \hline & -4x^3 + x^2(16 - k) - 25x + 10 \\ & -4x^3 + x^2(8) - 4xk \\ & + \quad - \quad + \\ \hline & x^2[8 - k] + x[4k - 25] + 10 \\ & x^2[8 - k] - 2x[8 - k] + k(8 - k) \\ & - \quad + \quad - \\ \hline & x[4k - 25 + 16 - 2k] + 10 - 8k + k^2 \end{array}$$

According to questions, remainder is x + a

$\therefore$ coefficient of x = 1

$\Rightarrow 2k - 9 = 1$

$\Rightarrow k = (10/2) = 5$

Also constant term = a

$\Rightarrow k^2 - 8k + 10 = a \Rightarrow (5)^2 - 8(5) + 10 = a$

$\Rightarrow a = 25 - 40 + 10$

$\Rightarrow a = -5$

$\therefore k = 5, a = -5$ **Ans.**